Hypergeometric-Gaussian Modes

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Abstract

We studied a novel family of paraxial laser beams forming an overcomplete yet nonorthogonal set of modes. These modes have a singular phase profile and are eigenfunctions of the photon orbital angular momentum. The intensity profile is characterized by a single brilliant ring with the singularity at its center, where the field amplitude vanishes. The complex amplitude is proportional to the degenerate (confluent) hypergeometric function, and therefore we term such beams hypergeometric gaussian (HyGG) modes. Unlike the recently introduced hypergeometric modes (Opt. Lett. **3**2, 742 (2007)), the HyGG modes carry a finite power and have been generated in this work with a liquid-crystal spatial light modulator. We briefly consider some sub-families of the HyGG modes as the modified Bessel Gaussian modes, the modified exponential Gaussian modes and the modified Laguerre-Gaussian modes.

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In the recent years there has been an increasing interest in laser beams especially tailored to experiments. In particular, experimentalists are looking for laser beams which are either nondiffracting or have a definite value of the photon orbital angular momentum (OAM) along the propagation direction. Such special laser beams have found useful applications in optical trapping, image processing, optical tweezers, metrology, microlithography, medical imaging and surgery, wireless and optical communications[1,2,3,4,5]. Moreover, beams carrying definite photon OAM present novel internal degrees of freedom that are potentially useful for quantum information applications[6,7]. Motivated by these issues, there has been an increasing interest in generating and studying laser beams corresponding to particular solutions of the scalar Helmholtz paraxial wave equation other than the well known Hermite-Gaussian (HG) and the Laguerre-Gaussian (LG) modes. Miller has solved the 3D Helmholtz paraxial wave equation in 17 coordinate systems, 11 of which were based on orthogonal coordinates[8]. Other examples of recently investigated paraxial beams are Parabolic, Mathieu, Ince-Gaussian, Helmholtz-Gaussian, Laplace-Gaussian, and pure light vortices (see[9] and references therein).

In this Letter we introduce a novel family of paraxial beams which are solutions of the scalar Helmoltz paraxial wave equation and are also eigenstates of the photon OAM. The field profile of these beams is proportional to the confluent hypergeometric function so we call them Hypergeometric-Gaussian modes (HyGG). Unlike the hypergeometric modes studied in[9], our HyGG modes carry a finite power so that they can be realized in practical experiments.
The field of the HyGG modes is given by

$$\begin{aligned} |HyGG\rangle_{pm} &= u_{pm}(\rho,\phi;\zeta) = C_{pm}\frac{\Gamma\left(1+|m|+\frac{p}{2}\right)}{\Gamma\left(|m|+1\right)} \\ &\quad \times i^{|m|+1}\zeta^{\frac{p}{2}}(\zeta+i)^{-(1+|m|+\frac{p}{2})} \\ &\quad \times \rho^{|m|}e^{-\frac{i\rho^2}{(\zeta+i)}}e^{im\phi} \\ &\quad \times {}_1F_1\left(-\frac{p}{2},|m|+1;\frac{\rho^2}{\zeta(\zeta+i)}\right) \end{aligned} \tag{1}$$

where m is integer, $p \geq -|m|$ is real valued, $\Gamma(x)$ is the gamma function and ${}_1F_1(a,b;x)$ is a confluent hypergeometric function[10]. In Eq. (1) we used dimensionless cylindrical coordinates $\rho = r/w_0$, ϕ, $\zeta = z/z_R$, where w_0 is the beam waist and $z_R = \pi w_0^2/\lambda$ is the beam Rayleigh range. We notice the characteristic factor $e^{im\phi}$ in the field, so that the integer values of m are identified with the eigenvalues of the photon OAM in units of $\hbar$. The normalization condition[11] ${}_{pm}\langle HyGG|HyGG\rangle_{pm} = 1$ fixes the constant C_{pm} to $C_{pm} = [2^{p+|m|+1}/\pi\Gamma(p+|m|+1)]^{1/2}$ up to a constant phase factor. The HyGG modes are an overcomplete not orthogonal set of modes. The inner product of two normalized HyGG modes is given by ${}_{p'm'}\langle HyGG|HyGG\rangle_{pm} = \delta_{mm'}\{\Gamma(p/2+p'/2+|m|+1)/[\Gamma(p'+|m|+1)\Gamma(p+|m|+1)]\}^{1/2}$. The asymptotic behavior of the intensity $|u_{pm}(\rho,\phi,\zeta)|^2$ of the HyGG modes as $\rho \to \infty$ at fixed $\zeta > 0$ is $|u_{pm}|^2 \propto \rho^{-2(2+p+|m|)}$ in general, and changes into the Gaussian behavior $|u_{pm}|^2 \propto \rho^{2(p+|m|)}\exp[-2\rho^2/(1+\zeta^2)]$ when p is a non negative even integer (see below). At the beam center ($\rho \to 0$) the field of the HyGG modes vanishes as $\rho^{|m|}$ as expected for the eigenmodes of the photon OAM (for $\zeta \geq 0$). Because all zeroes of the hypergeometric function ${}_1F_1(a,b;x)$ occur for real values of x, a characteristic feature of the HyGG modes given by Eq. (1) is that their intensity $|u_{pm}|^2$ never vanishes in the transverse plane, except at the beam axis $\rho = 0$ and at infinite. This confers to the intensity of the HyGG modes the typical doughnut shape (a single brilliant ring) for any values of m and p, except $p = m = 0$ when the HyGG mode reduces to the TEM_{00} Gaussian mode.
The limit of the field $u_{pm}(\rho,\phi;\zeta)$ at the pupil plane $\zeta \to 0^+$ is given by

$$\lim_{\zeta\to 0} u_{pm}(\rho,\phi;\zeta) = C_{pm}\,\rho^{p+|m|}e^{-\rho^2+im\phi}. \tag{2}$$

This feature is very useful because this pupil function can be generated by applying the singular phase factor $e^{im\phi}$ to a Gaussian-parabolic transmittance profile of the order $p+|m|$. In particular, the HyGG modes with $p = -|m|$ are simply generated by applying the phase factor $e^{im\phi}$ to a TEM_{00} laser field

at its waist plane.

The HyGG modes can be expanded in the complete basis of the Laguerre-Gauss (LG) modes. In general, the mode HyGG_{pm} is a superposition of the infinite LG_{qm} modes with same m and any integer $q \geq 0$. In fact, when both the HyGG and the LG modes are normalized, we have $|HyGG\rangle_{pm} = \sum_{q=0}^{\infty} A_{pq}|LG\rangle_{qm}$ with coefficients A_{pq} given by

$$A_{pq} = \sqrt{\frac{(q+|m|)!}{q!\,\Gamma(p+|m|+1)}}\,\frac{\Gamma(q-p/2)\Gamma(p/2+|m|+1)}{\Gamma(-p/2)\Gamma(q+|m|+1)}. \tag{3}$$

The HyGG modes can exhibit different features when the mode parameters p and m are changed. It is then convenient to separate the HyGG modes in a few subfamilies having similar properties.

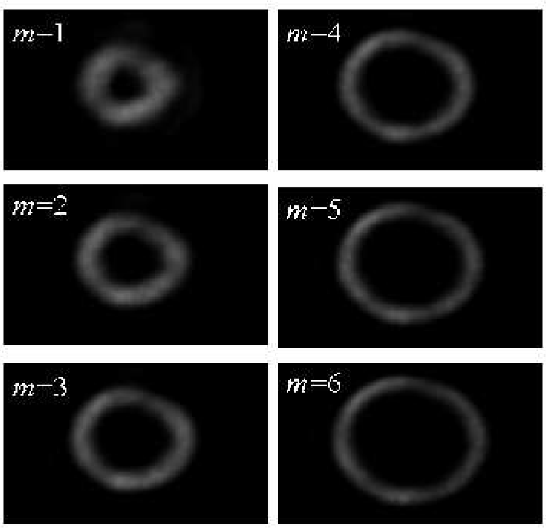

FIG. 1: Experimentally observed intensity distributions of the $|HyGG\rangle_{-m,m}$ mode for $m = 1,\ldots,6$ in the transverse plane $z = 0.18z_0$. The Rayleigh range was $z_0 = 747.4$ cm.

1. $p = m = 0$
   This mode is the well known Gaussian mode TEM_{00}.

2. $p = -|m|$, odd.
   The pupil field of these modes at $\zeta = 0$ is $\exp(-\rho^2 + im\phi)$. At planes $\zeta > 0$ the modes are linear combinations of the modified Bessel functions $I_0(x)$ and $I_1(x)$, where $x = \rho^2/2\zeta(\zeta+i)$. We may call this subfamily of modes the modified Bessel Gauss (MBG) modes. Unlike the well known Bessel modes, these modes carry a finite power and are not diffraction free. When $\rho \to \infty$ at fixed $\zeta > 0$, the intensity of these modes vanishes according to $|u_{pm}|^2 \propto \rho^{-4}$.

3. $p = -|m|$, even.
   The pupil field of these modes at $\zeta = 0$ is $\exp(-\rho^2 + im\phi)$. At planes $\zeta > 0$ the modes are linear combinations of exponential ρ-dependent terms. We may call this subfamily of modes the Modified Exponential Gauss (MEG) modes. When $\rho \to \infty$ at fixed ζ, the intensity of these modes vanishes according to $|u_{pm}|^2 \propto \rho^{-4}$.

4. $p \geq 0$, even.
   The pupil field of these modes is given by Eq. (2). When p is a non negative even integer, the confluent hypergeometric function reduces to a Laguerre polynomial. We will refer to these modes as to the modified Laguerre-Gauss modes (MLG). The asymptotic behaviour of the intensity of the MLG modes as $\rho \to \infty$ at fixed $\zeta > 0$ is the same as for the usual LG modes (i. e. $|u_{pm}|^2 \propto \rho^{2(p+|m|)}e^{-2\rho^2/(1+\zeta^2)}$). Unlike the LG_{pm} modes, however, the MLG_{pm} modes have a single-ring intensity profile for any admitted value of p. The MLG_{pm} modes can be expressed as the linear superposition of a finite number of LG_{qm} modes, namely, the LG_{qm} modes having the same m and integer $0 \leq q \leq p/2$. In fact, when p is a non negative even integer, Eq. (3) reduces to

$$A_{pq} = (-1)^q \frac{(p/2)!\,(p/2+|m|)!}{(p/2-q)!\sqrt{q!\,(p+|m|)!\,(q+|m|)!}} \tag{4}$$

   where $0 \leq q \leq p/2, p$ even. The quantities A_{pq} form the entries of a non singular $(p/2+1)\times(p/2+1)$ matrix. It is then obvious that this sub-family of HyGG modes forms a complete, yet not orthogonal, set of functions in the transverse plane and that the full set of HyGG modes is therefore overcomplete.

In our experiment, a TEM_{00} linearly polarized laser beam from a frequency doubled *Nd:YVO*$_4$ ($\lambda = 532$ nm, Model Verdi V5, Coherent) was used to illuminate a grey scale computer generated hologram (CGH) sent onto the LCD microdisplay of a spatial light modulator (SLM) (HoloEye Photonics LC-R 3000), with 1920×1200 pixels in a rectangle 18.24×11.40 mm wide. The SLM was located in the waist of the incident beam. We performed two series of measurements, according to the beam waist values w_0 at the SLM position. We measured w_0 by best fit between the error function and the integral curve of the intensity profile of incident beam as seen by CCD camera.The measured values were $w_0 = 1.13\pm0.04$ mm and $w_0 = 0.115\pm0.004$ mm, corresponding to Rayleigh ranges $z_0 = 747.4\pm50$ cm and $z_0 = 7.8\pm0.5$ cm, respectively. The theory was compared with the experimental data just using as parameters the measured values reported above, without further best fitting. We focused our attention on the HyGG modes with $p = -|m|$. In accordance with the theoretical predictions, we observed an intensity profile in the transverse plane essentially made of a single bright annulus, whatever the value of m we used or the observation z-plane. Some instances of the observed intensity profiles are shown in Fig. 1. The ring diameter of the beam as a function of m is reported in Fig 2. The diameter d of the ring was defined as the maximum distance between any two opposite maxima of the

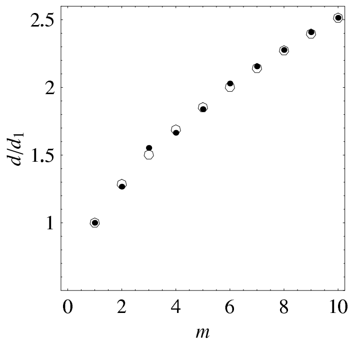

FIG. 2: The ring diameter d of the $|HyGG\rangle_{-m,m}$ mode for $m = 1,\ldots,10$ measured at plane $z = 0.18z_0$. The Rayleigh range was $z_0 = 747.4$ cm. The reported values of the diameters were scaled with respect to the value d_1 for $m = 1$. ○ – theory, ● – experiment.

intensity profile. The scaling law of d versus m turned to be in good agreement with the theoretical predictions. We measured also the ratio between the diameter $d(z)$ of the luminous ring and the gaussian beam size $w(z)$ at different z planes. The measurements were made by switching on and off the CGH to compare the intensity profile of the HyGG mode with the gaussian profile TEM_{00} beam profile at the same plane. Figure 3 shows that the ratio $d(z)/w(z)$ was the same in all z planes, as predicted by theory when $z > z_0$. The constant value of the ratio d/w obtained from the experiment was 3.1 ± 0.3, which is close to the theoretical prediction $d/w = 3.6$.

In summary, we studied a novel family of paraxial beams having hypergeometric field profile. This set of modes is overcomplete and nonorthogonal and all modes carry a finite power. In spite of their complicated field profile, these mode have a simple profile at the pupil plane. Finally we have experimentally produced these modes for different values of indexes. The agreement between experimental results and theoretical predictions turned to be satisfactory.

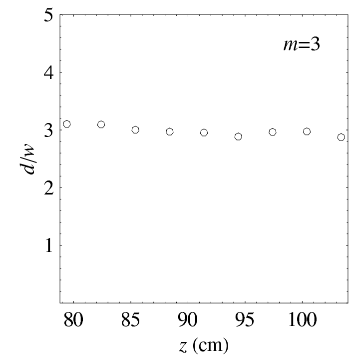

FIG. 3: The ratio between the diameter $d(z)$ of the $|HyGG\rangle_{-3,3}$ mode and the $1/e^2$ intensity radius $w(z)$ of the generating TEM_{00} gaussian beam as a function of z. The Rayleigh range was $z_0 = 7.8$ cm.

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[11] The Hilbert inner product is defined according to $\langle u|v\rangle = \int u^* v \rho d\rho d\phi$.